

A1) $\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^2 - x - 6} =$

A2) $\lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + 2}{5 - x^2 - 3x^4} =$

A3) $\lim_{x \rightarrow \infty} \sqrt{x} \sin\left(\frac{1}{\sqrt{x}}\right) =$

A4) The graph of $y = \frac{x}{1 + 25x^2}$ (for $x \geq 0$) has a horizontal tangent line at $x =$

A5) If $f(x) = \tan^{-1}(2x^2)$, then $f'\left(\frac{1}{2}\right) =$

A6) If $f(x) = \frac{5x-1}{2-3x}$, then $f^{-1}(x) =$

A7) Let $f(x) = \begin{cases} x^2 - c^2 & \text{if } x < 3 \\ cx - 1 & \text{if } x \geq 3 \end{cases}$, where c is a negative constant. Find the value of c so that f is continuous everywhere.A8) Let $f(x) = |x^3 - 5x - 4|$. Suppose that g is a differentiable function such that $g(8) = 2$. Let $h(x) = f[g(f(x))]$. If $h'(3) = 14$, what is the value of $g'(8)$?A9) Let $f^{(n)}(a)$ denote the n -th derivative of f at a . If $f(x) = \sin^4 x - \cos^4 x$, then $f^{(73)}\left(\frac{\pi}{12}\right) =$ A10) The orthogonal trajectories of the family of curves $x^2 - y^2 = k$ are the curves

- B1) Let $f(x) = x^2$. Find $f'(x)$ from first principles (i.e. by using only the definition of the derivative).
- B2) Use any suitable method to find $\frac{dy}{dx}$ for each of the following. There is no need to simplify your final answers for the question.
- a) $y = e^{2x} \sin 3x$
- b) $y = \frac{\tan x}{1 + x^2}$
- B3) For this question, simplify your final answers as much as possible.
- a) If $f(x) = \sqrt{2 + \sqrt{x}}$, find $f'(4)$.
- b) If $f(x) = 2^{\frac{1}{x}}$, find $f'\left(\frac{1}{2}\right)$.
- B4) Suppose that $xy^3 + 2y^2 + xy + 2x^3 = 0$. Find the general expression for $\frac{dy}{dx}$. Also, find the value of $\frac{dy}{dx}$ at the point where $x = 1$.
- B5) A ball is being thrown upward from the top of a building so that its height (above ground) t seconds after it is thrown is $90 + 64t - 16t^2$ feet. Find the velocity of the ball when it is 10 feet above ground on its way down.
- B6) Find the line passing through the point $\left(0, -\frac{3}{4}\right)$ and tangent to the curve $y = 4x^4$ (where $x \geq 0$) at some point.
- B7) Let $f(x) = \begin{cases} (x+3)^2 & \text{if } x < 0 \\ -x^2 + 8x - 4 & \text{if } x \geq 0 \end{cases}$. Find the line which is tangent to the graph of f at two points.

$$A1) \lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^2 - x - 6} =$$

$$\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^2 - x - 6}$$

$$= \frac{3^2 - 2 \cdot 3 - 3}{3^2 - 3 - 6}$$

$$= \frac{0}{0}$$

- Let's first see if this is defined by substituting in the limit.

- Now we evaluate and simplify.

- The limit is not defined. We therefore need an alternate way to determine the limit.

$$\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^2 - x - 6}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{(x-3)(x+2)}$$

$$= \lim_{x \rightarrow 3} \frac{x+1}{x+2}$$

$$= \frac{3+1}{3+2}$$

$$= \frac{4}{5}$$

- Since both the numerator and denominator are polynomials, let's factorize them.

- Both numerator and denominator has the factor $x = 3$, which can be cancelled out.

- This cannot be simplified further, so we substitute in the limit.

- Last step is to evaluate this expression.

- This is a defined limit, and the answer.

$$A2) \lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + 2}{5 - x^2 - 3x^4} =$$

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + 2}{5 - x^2 - 3x^4} \\ &= \frac{\infty^4 - 2 \cdot \infty^3 + 2}{5 - \infty^2 - 3 \cdot \infty^4} \\ &= \frac{\infty}{\infty} \end{aligned}$$

- Let's first see if this is defined by substituting in the limit.
- ∞ raised to any power is still ∞ , and subtracting any number from ∞ is also still ∞ .
- The limit is not defined. We therefore need an alternate way to determine the limit..

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + 2}{5 - x^2 - 3x^4} \\ &= \lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + 2}{5 - x^2 - 3x^4} \cdot \frac{1}{x^4} \end{aligned}$$

- Let's divide top and bottom by x^4 , the largest power of x , since $c/\infty = 0$ for any number c .

$$= \lim_{x \rightarrow \infty} \frac{1 - \frac{2}{x} + \frac{2}{x^4}}{\frac{5}{x^4} - \frac{1}{x^2} - 3}$$

- Now we simplify the expression.

$$\begin{aligned} &= \frac{1 - \frac{2}{\infty} + \frac{2}{\infty^4}}{\frac{5}{\infty^4} - \frac{1}{\infty^2} - 3} \\ &= \frac{1}{-3} \end{aligned}$$

- This cannot be simplified further, so we substitute in the limit.

- As we noted before, ∞ raised to any power is still ∞ , and $c/\infty = 0$, which means 4 of the terms are 0.

- This is a defined limit, and the answer.

$$A3) \lim_{x \rightarrow \infty} \sqrt{x} \sin\left(\frac{1}{\sqrt{x}}\right) =$$

$$\lim_{x \rightarrow \infty} \sqrt{x} \sin\left(\frac{1}{\sqrt{x}}\right)$$

- Let's first see if this is defined by substituting in the limit.

$$= \sqrt{\infty} \sin\left(\frac{1}{\sqrt{\infty}}\right)$$

- $\sqrt{\infty}$ is still ∞ , and $1/\infty = 0$.

$$= \infty \cdot \sin(0)$$

- $\sin(0) = 0$.

$$= \infty \cdot 0$$

- The limit is not defined. We therefore need an alternate way to determine the limit.

$$\lim_{x \rightarrow \infty} \sqrt{x} \sin\left(\frac{1}{\sqrt{x}}\right)$$

- Thinking towards l'Hopital's Rule, let's divide top and bottom by \sqrt{x} to get a fraction.

$$= \lim_{x \rightarrow \infty} \sqrt{x} \sin\left(\frac{1}{\sqrt{x}}\right) \cdot \frac{1}{\sqrt{x}}$$

- Now we simplify the expression.

$$= \lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{\sqrt{x}}\right)}{\frac{1}{\sqrt{x}}}$$

Since $1/\sqrt{\infty} = 0$, we now have a limit of the form $0/0$, so we can apply l'Hopital's Rule. However, to simplify the algebra, we can make the substitution $u = 1/\sqrt{x}$, with the limit $u \rightarrow 0^+$.

$$= \lim_{u \rightarrow 0^+} \frac{\sin u}{u}$$

- Now we apply l'Hopital's Rule: take the derivative top and bottom individually.

$$= \lim_{u \rightarrow 0^+} \frac{\frac{d}{du} \sin u}{\frac{d}{du} u}$$

- We evaluate these derivatives.

$$= \lim_{u \rightarrow 0^+} \frac{\cos u}{1}$$

- This cannot be simplified further, so we substitute in the limit.

$$= \cos(0)$$

$$= \underline{1}$$

- This is a defined limit, and the answer.

A4) The graph of $y = \frac{x}{1+25x^2}$ (for $x \geq 0$) has a horizontal tangent line at $x =$

A horizontal tangent line occurs when $\frac{dy}{dx} = 0$, therefore:

$$\frac{d}{dx} \left(\frac{x}{1+25x^2} \right) = 0 \quad - \text{ Take the derivative using the Quotient Rule.}$$

$$\frac{\left(\frac{d}{dx} x \right) \cdot (1+25x^2) - x \cdot \frac{d}{dx} (1+25x^2)}{(1+25x^2)^2} = 0 \quad - \text{ We next take the required derivatives.}$$

$$\frac{1 \cdot (1+25x^2) - x \cdot 50x}{(1+25x^2)^2} = 0 \quad - \text{ Simplifying and grouping terms is next.}$$

$$\frac{1-25x^2}{(1+25x^2)^2} = 0 \quad - \text{ The only way to get a fraction to be equal 0 is if the numerator is 0.}$$

$$1-25x^2 = 0$$

$$25x^2 = 1 \quad - \text{ Simplify and solve for } x.$$

$$x^2 = \frac{1}{25}$$

$$x = \frac{1}{5} \quad - \text{ Since } x \geq 0, \text{ we discard the negative answer.}$$

A5) If $f(x) = \tan^{-1}(2x^2)$, then $f'\left(\frac{1}{2}\right) =$

$$f'(x) = \frac{d}{dx} [\tan^{-1}(2x^2)]$$

- Take the derivative, keeping in mind that we need to invoke the Chain Rule.

$$= \frac{1}{1+(2x^2)^2} \cdot \frac{d}{dx} 2x^2$$

- Take the derivative and simplify.

$$= \frac{4x}{1+4x^4}$$

- This is the expression we substitute in $x = \frac{1}{2}$.

$$f'\left(\frac{1}{2}\right) = \frac{4\left(\frac{1}{2}\right)}{1+4\left(\frac{1}{2}\right)^4}$$

- Replace all occurrences of x with $\frac{1}{2}$.

$$= \frac{2}{1+\frac{1}{4}}$$

- Simplify and arrive at the answer.

$$= 2 \cdot \frac{4}{5}$$

$$= \frac{8}{5}$$

A6) If $f(x) = \frac{5x-1}{2-3x}$, then $f^{-1}(x) =$

To simplify the notation, let's write instead $y = \frac{5x-1}{2-3x}$.

$$x = \frac{5y-1}{2-3y}$$

$$x(2-3y) = 5y-1$$

$$2x - 3xy = 5y - 1$$

$$5y + 3xy = 2x + 1$$

$$y(6x+3) = 2x+1$$

$$y = \frac{2x+1}{\underline{\underline{3x+3}}}$$

We first write out the expression swapping x and y . The goal is to solve for y , which is $f^{-1}(x)$.

- Multiply both sides by the denominator.

- Expand the left hand side.

- Move only terms containing y to the left hand side.

- Factorize out y .

- Divide both sides by $6x+3$ and we arrive at the answer.

A7) Let $f(x) = \begin{cases} x^2 - c^2 & \text{if } x < 3 \\ cx - 1 & \text{if } x \geq 3 \end{cases}$, where c is a negative constant. Find the value of c so that f is continuous everywhere.

To have f continuous everywhere means that it must be the same value using either expression where the distinction occurs at $x = 3$:

$$x^2 - c^2 = cx - 1 \quad - \text{ This is required for a function to be continuous.}$$

$$3^2 - c^2 = c(3) - 1 \quad - \text{ Substitute in } x = 3.$$

$$c^2 + 3c - 10 = 0 \quad - \text{ Move all terms to the left and collect terms.}$$

$$(c + 5)(c - 3) = 0 \quad - \text{ Factorize.}$$

$$c = \underline{\underline{-5}} \quad - \text{ We discard the positive root as asked in question.}$$

A8) Let $f(x) = |x^3 - 5x - 4|$. Suppose that g is a differentiable function such that $g(8) = 2$. Let

$h(x) = f[g(f(x))]$. If $h'(3) = 14$, what is the value of $g'(8)$?

$$h'(x) = \frac{d}{dx} f(g[f(x)])$$

- Evaluate the derivative of $h(x)$ using the Chain Rule.

$$= f'(g[f(x)]) \cdot \frac{d}{dx} g[f(x)]$$

- Chain Rule is needed again to evaluate the derivative.

$$= f'(g[f(x)]) \underbrace{g'[f(x)] \cdot f'(x)}_{\text{Chain Rule}}$$

$$= f'(g[f(x)]) g'[f(x)] f'(x)$$

- This is the expression for $h'(x)$.

$$h'(3) = 14$$

- This is given.

$$f'(g[f(3)]) g'[f(3)] f'(3) = 14$$

- Substitute in expression for $h'(x)$.

$$f'[g(8)] g'(8) \cdot (3x^2 - 5) \Big|_{x=3} = 14$$

- Substitute in $f(3) = |3^3 - 5 \cdot 3 - 4| = 8$ and expression for $f'(x): f'(x) = x^3 - 5x - 4$ since $f(3) > 0$.

$$f'(2) g'(8) \cdot (3 \cdot 3^2 - 5) = 14$$

- Substitute in $g(8) = 2$ and $x = 3$.

$$-(3x^2 - 5) \Big|_{x=2} g'(8) \cdot 22 = 14$$

- Substitute in expression for $f'(x)$: note that $f(2) = |-6|$, such that $f'(x) = -(x^3 - 5x - 4)$.

$$-(3 \cdot 2^2 - 5) g'(8) \cdot 22 = 14$$

- Substitute in $x = 2$.

$$-7 g'(8) \cdot 22 = 14$$

- Simplify.

$$g'(8) = -\frac{14}{7 \cdot 22}$$

- Divide both sides by constants.

$$= -\frac{1}{11}$$

- Reduce the fraction and arrive at the answer.

A9) Let $f^{(n)}(a)$ denote the n -th derivative of f at a . If $f(x) = \sin^4 x - \cos^4 x$, then $f^{(73)}\left(\frac{\pi}{12}\right) =$

$$\begin{aligned} f(x) &= \sin^4 x - \cos^4 x && - \text{We need to first rewrite/simplify } f. \\ &= (\sin^2 x)^2 - (\cos^2 x)^2 && - \text{Rewrite to difference of squares.} \\ &= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) && - a^2 - b^2 = (a - b)(a + b). \\ &= -(\cos^2 x - \sin^2 x) \cdot 1 && - \text{Substitute in identity } \sin^2 x + \cos^2 x = 1. \\ &= -\cos 2x && - \text{Substitute in identity } \cos 2x = \cos^2 x - \sin^2 x. \end{aligned}$$

$$\begin{aligned} f^{(1)}(x) &= \frac{d}{dx}[-\cos(2x)] && - \text{We now take the 1}^{\text{st}} \text{ derivative using the Chain Rule.} \\ &= 2\sin(2x) && - \text{This is the expression for the 1}^{\text{st}} \text{ derivative.} \end{aligned}$$

$$\begin{aligned} f^{(2)}(x) &= \frac{d}{dx} 2\sin(2x) && - \text{We now take the 2}^{\text{nd}} \text{ derivative using the Chain Rule.} \\ &= 2^2 \cos(2x) && - \text{This is the expression for the 2}^{\text{nd}} \text{ derivative.} \end{aligned}$$

$$\begin{aligned} f^{(3)}(x) &= \frac{d}{dx} 2^2 \cos(2x) && - \text{We now take the 3}^{\text{rd}} \text{ derivative using the Chain Rule.} \\ &= -2^3 \sin(2x) && - \text{This is the expression for the 3}^{\text{rd}} \text{ derivative.} \end{aligned}$$

$$\begin{aligned} f^{(4)}(x) &= \frac{d}{dx}[-2^3 \sin(2x)] && - \text{We now take the 4}^{\text{th}} \text{ derivative using the Chain Rule.} \\ &= -2^4 \cos(2x) && - \text{This is the expression for the 4}^{\text{th}} \text{ derivative.} \end{aligned}$$

A pattern has emerged: starting from the 1st derivative being positive, the sign changes every 2 derivatives. As well, taking the derivative an odd number of times yields $\sin(2x)$, while taking the derivative an even number of times yields $\cos(2x)$. Lastly, there is a factor of 2 raised to the number of times the derivative is taken. Mathematically then:

$$f^{(n)}(x) = \begin{cases} (-1)^{\frac{n-1}{2}} 2^n \sin(2x) & \text{if } n \text{ is odd} \\ (-1)^{\frac{n+2}{2}} 2^n \cos(2x) & \text{if } n \text{ is even} \end{cases}$$

$$\begin{aligned} f^{(73)}\left(\frac{\pi}{12}\right) &= (-1)^{\frac{73-1}{2}} 2^{73} \sin\left(2 \cdot \frac{\pi}{12}\right) && - \text{Substitute } x = \pi/12 \text{ and } n = 73 \text{ into 1}^{\text{st}} \text{ expression since 73 is odd.} \\ &= (-1)^{36} 2^{73} \sin \frac{\pi}{6} && - \text{Simplify the arithmetic.} \\ &= 2^{73} \cdot \frac{1}{2} && - \text{Further simplification.} \\ &= \underline{\underline{2^{72}}} && - \text{This is the answer.} \end{aligned}$$

A10) The orthogonal trajectories of the family of curves $x^2 - y^2 = k$ are the curves

Orthogonal trajectories require that $slope_1 = -\frac{1}{slope_2}$ where the curves intersect.

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}k \quad - \text{ We first evaluate the slope of the given family of curves.}$$

$$2x - \underbrace{2y \frac{dy}{dx}}_{\text{Chain Rule}} = 0 \quad - \text{ Chain Rule is required for the 2}^{\text{nd}} \text{ term.}$$

$$2y \frac{dy}{dx} = 2x \quad - \text{ Subtract both sides by the } y\text{-containing term.}$$

$$\frac{dy}{dx} = \frac{x}{y} \quad - \text{ Divide both sides by } 2y.$$

The slope of the family of curves that has orthogonal trajectories is therefore:

$$\frac{dy}{dx} = -\frac{y}{x} \quad - \text{ This is the slope of the family of curves we seek.}$$

$$x \frac{dy}{dx} = -y \quad - \text{ Multiply both sides by } x.$$

$$y + x \frac{dy}{dx} = 0 \quad - \text{ Add } y \text{ to both sides.}$$

$$\frac{dx}{dx} y + x \frac{dy}{dx} = 0 \quad - \text{ Multiplying } dx/dx = 1 \text{ to the first term to yield an expression corresponding to the Product Rule.}$$

$$\frac{d}{dx}(xy) = 0 \quad - \text{ Substituting in result of Product Rule.}$$

Thus, the family of curves we seek is:

$$\underline{\underline{xy = C}}$$

B1) Let $f(x) = x^2$. Find $f'(x)$ from first principles (i.e. by using only the definition of the derivative).

$$\begin{aligned} f'(x) &= \lim_{a \rightarrow x} \frac{f(a) - f(x)}{a - x} && - \text{ This is the definition of the derivative.} \\ &= \lim_{a \rightarrow x} \frac{a^2 - x^2}{a - x} && - \text{ Substitute in the function } f(x) = x^2. \\ &= \lim_{a \rightarrow x} \frac{(a - x) \cdot (a + x)}{a - x} && - \text{ Factorize the numerator using the fact that it's the difference of squares.} \\ &= \lim_{a \rightarrow x} (a + x) && - \text{ Cancel out the common factor } (a - x). \\ &= x + x && - \text{ Substitute in the limit.} \\ &= \underline{\underline{2x}} && - \text{ This is the answer, as should be.} \end{aligned}$$

B2) Use any suitable method to find $\frac{dy}{dx}$ for each of the following. There is no need to simplify your final answers for the question.

a) $y = e^{2x} \sin 3x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(e^{2x} \sin 3x) && \text{- Apply Product Rule.} \\ &= \left(\frac{d}{dx} e^{2x}\right) \cdot \sin 3x + e^{2x} \cdot \frac{d}{dx} \sin 3x && \text{- Apply Chain Rule on these subsequent derivatives.} \\ &= e^{2x} \cdot \underbrace{\frac{d}{dx}(2x)}_{\text{Chain Rule}} \sin 3x + e^{2x} \cos 3x \cdot \underbrace{\frac{d}{dx}(3x)}_{\text{Chain Rule}} && \text{- Simplify.} \\ &= \underline{\underline{e^{2x} \sin 3x + 3e^{2x} \cos 3x}} \end{aligned}$$

b) $y = \frac{\tan x}{1+x^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\tan x}{1+x^2} \right) && \text{- Use the Quotient Rule.} \\ &= \frac{\left(\frac{d}{dx} \tan x\right) \cdot (1+x^2) - \tan x \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2} \\ &= \underline{\underline{\frac{\sec^2 x(1+x^2) - \tan x \cdot 2x}{(1+x^2)^2}}} \end{aligned}$$

B3) For this question, simplify your final answers as much as possible.

a) If $f(x) = \sqrt{2 + \sqrt{x}}$, find $f'(4)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx} \sqrt{2 + \sqrt{x}} && \text{- Apply Chain Rule.} \\ &= \frac{1}{2\sqrt{2 + \sqrt{x}}} \cdot \frac{d}{dx} (2 + \sqrt{x}) \\ &= \frac{1}{2\sqrt{2 + \sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} \end{aligned}$$

$$\begin{aligned} f'(4) &= \frac{1}{2\sqrt{2 + \sqrt{4}}} \cdot \frac{1}{2\sqrt{4}} && \text{- Substitute in } x = 4. \\ &= \frac{1}{2\sqrt{4}} \cdot \frac{1}{4} && \text{- Simplify.} \\ &= \frac{1}{4} \cdot \frac{1}{4} && \text{- Further Simplify.} \\ &= \frac{1}{16} \end{aligned}$$

b) If $f(x) = 2^{\frac{1}{x}}$, find $f'\left(\frac{1}{2}\right)$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} 2^{\frac{1}{x}} && \text{- Apply Chain Rule.} \\ &= \left(2^{\frac{1}{x}} \ln 2\right) \cdot \frac{d}{dx} \frac{1}{x} \\ &= \left(2^{\frac{1}{x}} \ln 2\right) \cdot \frac{-1}{x^2} \end{aligned}$$

$$\begin{aligned} f'\left(\frac{1}{2}\right) &= \left(2^{\left(\frac{1}{\frac{1}{2}}\right)} \ln 2\right) \cdot \frac{-1}{\left(\frac{1}{2}\right)^2} && \text{- Substitute in } x = \frac{1}{2}. \\ &= -2^2 \ln 2 \cdot 4 && \text{- Simplify.} \\ &= \underline{\underline{-16 \ln 2}} \end{aligned}$$

B4) Suppose that $xy^3 + 2y^2 + xy + 2x^3 = 0$. Find the general expression for $\frac{dy}{dx}$. Also, find the value of $\frac{dy}{dx}$ at the point where $x = 1$.

$$\frac{d}{dx}(xy^3 + 2y^2 + xy + 2x^3) = \frac{d}{dx}0 \quad - \text{Differentiate of both sides.}$$

$$\underbrace{\frac{dx}{dx}y^3 + x \frac{d}{dx}y^3}_{\text{Product Rule}} + \underbrace{4y \frac{dy}{dx}}_{\text{Chain Rule}} + \underbrace{\frac{dx}{dx}y + x \frac{dy}{dx}}_{\text{Product Rule}} + 6x^2 = 0 \quad - \text{Apply various rules.}$$

$$y^3 + x \cdot 3y^2 \frac{dy}{dx} + 4y \frac{dy}{dx} + y + x \frac{dy}{dx} + 6x^2 = 0 \quad - \text{Apply Chain Rule.}$$

Chain Rule

$$3xy^2 \frac{dy}{dx} + 4y \frac{dy}{dx} + x \frac{dy}{dx} = -y^3 - y - 6x^2 \quad - \text{Move all } dy/dx \text{ terms to one side.}$$

$$(3xy^2 + 4y + x) \frac{dy}{dx} = -(y^3 + y + 6x^2) \quad - \text{Factorize out } dy/dx.$$

$$\frac{dy}{dx} = \underline{\underline{-\frac{y^3 + y + 6x^2}{3xy^2 + 4y + x}}}$$

$$(xy^3 + 2y^2 + xy + 2x^3) \Big|_{x=1} = 0 \quad - \text{Find the corresponding } y \text{ value when } x = 1.$$

$$1 \cdot y^3 + 2y^2 + 1 \cdot y + 2 \cdot 1^3 = 0 \quad - \text{Substitute in } x = 1.$$

$$y^3 + 2y^2 + y + 2 = 0 \quad - y = -2 \text{ yields } y^3 + 2y^2 + y + 2 = 0, \text{ so } (y + 2) \text{ is a factor.}$$

- Take out $(y + 2)$ as a factor.

$$(y + 2)(y^2 + 0y + 1) = 0 \quad - \text{Discard } y^2 + 1 = 0 \text{ since root is imaginary.}$$

$$y = -2$$

$$\frac{dy}{dx} \Big|_{x=1, y=-2} = -\frac{(-2)^3 + (-2) + 6 \cdot 1^2}{3 \cdot 1 \cdot (-2)^2 + 4(-2) + 1} \quad - \text{We now evaluate } dy/dx \text{ at the point } (1, -2).$$

$$= -\frac{-8 - 2 + 6}{12 - 8 + 1}$$

$$= -\frac{-4}{5}$$

$$= \underline{\underline{\frac{4}{5}}}$$

B5) A ball is being thrown upward from the top of a building so that its height (above ground) t seconds after it is thrown is $90 + 64t - 16t^2$ feet. Find the velocity of the ball when it is 10 feet above ground on its way down.

Let the height above ground be $h(t) = 90 + 64t - 16t^2$. The velocity is then the derivative $h'(t)$:

$$h'(t) = \frac{d}{dt}(90 + 64t - 16t^2) \quad - \text{ Take the derivative for an expression of velocity.}$$

$$= 64 - 32t$$

$$h(t) = 10 \quad - \text{ Find the time corresponding to a height of 10 feet.}$$

$$90 + 64t - 16t^2 = 10 \quad - \text{ Substitute in expression for } h(t).$$

$$16t^2 - 64t - 80 = 0 \quad - \text{ Multiply both sides by } -1.$$

$$t^2 - 4t - 5 = 0 \quad - \text{ Divide both sides by } 16.$$

$$(t - 5)(t + 1) = 0 \quad - \text{ Factorize.}$$

$$t = 5 \quad - \text{ Discard } t = -1 \text{ since negative time is meaningless.}$$

$$h'(5) = 64 - 32 \cdot 5 \quad - \text{ Substitute in } t = 5.$$

$$= 64 - 160$$

$$= \underline{\underline{-96}} \quad - \text{ The unit is "feet per second".}$$

B6) Find the line passing through the point $\left(0, -\frac{3}{4}\right)$ and tangent to the curve $y = 4x^4$ (where $x \geq 0$) at some point.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} 4x^4 && \text{- The slope of tangent line at any point along curve } y = 4x^4 \text{ is its derivative.} \\ &= 16x^3\end{aligned}$$

If we let the point that forms a tangent line with $\left(0, -\frac{3}{4}\right)$ be $(a, b) = (a, 4a^4)$:

$$\frac{4a^4 - \left(-\frac{3}{4}\right)}{a - 0} = \left. \frac{dy}{dx} \right|_{x=a} \quad \text{- The slope between the two points must be equal to the slope of the tangent line.}$$

$$\frac{4a^4 + \frac{3}{4}}{a} = 16a^3 \quad \text{- Substitute in the slope of the tangent line at } x = a.$$

$$4a^4 + \frac{3}{4} = 16a^4 \quad \text{- Multiply both sides by } a.$$

$$12a^4 = \frac{3}{4} \quad \text{- Collect all } a^4 \text{ terms on one side.}$$

$$a^4 = \frac{3}{4 \cdot 12} \quad \text{- Divide both sides by 12.}$$

$$= \frac{1}{16}$$

$$a = \frac{1}{2}$$

$$\frac{y - \left(-\frac{3}{4}\right)}{x - 0} = 16a^3 \quad \text{- General equation of line: } \frac{y - y_1}{x - x_1} = \text{slope} \text{ if } (x_1, y_1) \text{ is on the line.}$$

$$\frac{y + \frac{3}{4}}{x} = 16 \cdot \left(\frac{1}{2}\right)^3 \quad \text{- Substitute in } a = \frac{1}{2}.$$

$$\frac{y + \frac{3}{4}}{x} = 2$$

$$y + \frac{3}{4} = 2x \quad \text{- Multiply both sides by } x.$$

$$y = 2x - \frac{3}{4} \quad \text{- Subtract constant from both sides.}$$

The tangent line is therefore $y = 2x - \frac{3}{4}$.

B7) Let $f(x) = \begin{cases} (x+3)^2 & \text{if } x < 0 \\ -x^2 + 8x - 4 & \text{if } x \geq 0 \end{cases}$. Find the line which is tangent to the graph of f at two points.

Since both expressions are quadratic, the two points cannot lie on the same side of the y -axis.

$$\text{Let } g(x) = (x+3)^2, \quad h(x) = -x^2 + 8x - 4$$

$$\begin{aligned} g'(x) &= \frac{d}{dx}(x+3)^2 && \text{- Slope of tangent line is the derivative, so we first take the derivative of } f \text{ for} \\ &= 2(x+3) && \text{negative } x. \\ &= 2x + 6 \end{aligned}$$

$$\begin{aligned} h'(x) &= \frac{d}{dx}(-x^2 + 8x - 4) && \text{- We next take the derivative of } f \text{ for positive } x. \\ &= -2x + 8 \end{aligned}$$

Let points $(x_1, y_1) = (x_1, g(x_1))$ where $x_1 < 0$ and $(x_2, y_2) = (x_2, h(x_2))$ where $x_2 \geq 0$ be points on the curve $y = f(x)$. We now have a set of simultaneous equations we need to solve:

$$\frac{y_2 - y_1}{x_2 - x_1} = g'(x_1) \quad \text{Eqn. 1} \quad \text{- The slope calculated using the 2 points is the same as the derivative at each of the points.}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = h'(x_2) \quad \text{Eqn. 2} \quad \text{- The slope calculated using the 2 points is the same as the derivative at each of the points.}$$

$$g'(x_1) = h'(x_2) \quad \text{Eqn. 3} \quad \text{- Left hand side of Eqn. 1 and 2 are the same.}$$

$$2x_1 + 6 = -2x_2 + 8 \quad \text{- Substitute the derivatives into Eqn. 3.}$$

$$2x_2 = 8 - 6 - 2x_1 \quad \text{- Move } x_2 \text{ term to one side.}$$

$$x_2 = 1 - x_1 \quad \text{Eqn. 3'} \quad \text{- Divide both sides by 2.}$$

$$\frac{h(x_2) - g(x_1)}{x_2 - x_1} = 2x_1 + 6$$

- Sub y-coordinate of the 2 points into Eqn. 1.

$$\frac{(-x_2^2 + 8x_2 - 4) - (x_1 + 3)^2}{x_2 - x_1} = 2x_1 + 6$$

- Sub in the functions.

$$\frac{[-(1-x_1)^2 + 8(1-x_1) - 4] - (x_1 + 3)^2}{(1-x_1) - x_1} = 2x_1 + 6$$

- Sub in Eqn. 3' for x_2 .

$$\frac{-(1-2x_1+x_1^2) + 8-8x_1-4 - (x_1^2+6x_1+9)}{1-2x_1} = 2x_1 + 6$$

- Expand the terms.

$$\frac{-x_1^2 - x_1^2 + 2x_1 - 8x_1 - 6x_1 - 1 + 8 - 4 - 9}{1-2x_1} = 2x_1 + 6$$

- Group terms.

$$\frac{-2x_1^2 - 12x_1 - 6}{1-2x_1} = 2x_1 + 6$$

- Simplify numerator.

$$2(-x_1^2 - 6x_1 - 3) = 2(x_1 + 3)(1-2x_1)$$

- Simplify.

$$-x_1^2 - 6x_1 - 3 = -2x_1^2 - 5x_1 + 3$$

$$x_1^2 - x_1 - 6 = 0$$

- Factorize.

$$(x_1 - 3)(x_1 + 2) = 0$$

- Since $x_1 < 0$.

$$x_1 = -2$$

$$\frac{y - g(x_1)}{x - x_1} = g'(x_1)$$

- General equation of a line: $\frac{y - y_1}{x - x_1} = \text{slope}$, with $(x_1, y_1) = (x_1, g(x_1))$ and the slope at this point.

$$\frac{y - g(-2)}{x - (-2)} = g'(-2)$$

- Substitute in $x_1 = -2$.

$$\frac{y - (-2+3)^2}{x - (-2)} = 2(-2) + 6$$

- Substitute in the expression for $g(x)$ and $g'(x)$.

$$\frac{y-1}{x+2} = 2$$

- Simplify.

$$y-1 = 2(x+2)$$

$$y = 2x + 4 + 1$$

$$= 2x + 5$$

Therefore, the line we seek is $y = 2x + 5$