A1) Find the value of $\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^3-x^2-6x}$

A2) Find the value of $\lim_{x\to 0}\left(\frac{\sin 5x}{\sin 2x}-\frac{\sin 3x}{4x}\right)$.

A3) Find the value of $\lim_{x\to\infty}\frac{3x^4-x^3+4}{5+x^2-8x^4}$.

A4) If $f(x)=\frac{2x-3}{5x+4}$, then $f^{-1}(x)=$

A5) If $f(x)=\sin^{-1}\left(\frac{x}{3}\right)$, then $f'\left(\sqrt{5}\right)=$

A6) If $2x^2-xy+y^3=20$, find the value of $\frac{dy}{dx}$ at the point where $x=3$, $y=2$.

A7) Let $f(x)=ax^2+bx$, where a and b are constants. If the tangent line to the curve $y=f(x)$ at the point $(1,1)$ has the equation $y=3x-2$, then $f(3)=$

A8) If m is a fixed constant such that the two families of curves $y=\frac{1}{x+c}$ and $y=m(x+k)^{\frac{1}{3}}$ are orthogonal trajectories of each other, what must be the value of m?

A9) Let $h(x)=\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials. Consider the following three statements.

I. If $\lim_{x\to 5}g(x)=g(5)$ and h is continues at 5, then f must be continuous at 5.

II. If $(x-4)$ is a factor of $g(x)$, then the line $x=4$ must be a vertical asymptote of the curve $y=h(x)$.

III. If $g(3)<0$ and $\frac{g(3)}{g'(3)}=\frac{f(3)}{f'(3)}$, then the curve $y=h(x)$ must have a horizontal tangent line at $x=3$.

A10) If $y^2+2e^{-xy}=6$, find the value of $\frac{d^2y}{dx^2}$ at the point where $y=2$.

B1) Let $f(x)=\sqrt{x}$. Find $f'(x)$ from first principles (i.e. by using only the definition of the derivative).

B2) Use any suitable method to find $\frac{dy}{dx}$ for each of the following. There is no need to simplify your final answers for the question.

a) $y=x^2e^{3x}$

b) $y=\frac{2+x^2}{(3+x)^5}$

c) $y=\tan(\sqrt{x})$

d) $y=5^{\tan^{-1}(x^2)}$

B3) Find the line which passes through the point $\left(0,\frac{1}{4}\right)$ and is tangent to the curve $y=x^3$ at some point.

B4) Let $f(x)=\begin{cases} 2x^3 & \text{if } x<2 \\ x^2+ax+b & \text{if } -2<x<2 \\ 3x^4-48 & \text{if } x\geq 2 \end{cases}$. Find the values of the constants a and b so that f is continuous everywhere.

B5) An object is being thrown upward so that its height (above ground) t seconds after it is thrown is $100+48t-16t^2$ feet. Find the velocity of the object when it is 132 feet above ground on its way up.

B6) Evaluate $\lim_{x\to\infty}\left(\sqrt[3]{x^3+5x^2}-x\right)$. Remember to fully justify your answer.

A1) Find the value of $\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^3-x^2-6x}$

i) Try substitute into the limit.

$$\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^3-x^2-6x}=\frac{(3)^3-2(3)^2-3(3)}{(3)^3-(3)^2-6(3)}$$ - Substitute 3 for x into the limit.

$$=\frac{27-18-9}{27-9-18}$$ - Evaluate.

$$=\frac{0}{0}$$ - $\frac{0}{0}$ is undefined, go to and try factoring.

The limit is undefined, so go to step ii.

ii) Factor the numerator and the denominator. Cancel out factors common to the numerator and denominator. Go to step i.

$$\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^3-x^2-6x}=\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^3-x^2-6x}$$ - Take Common Factor out of numerator and the denominator.

$$=\lim_{x\to 3}\frac{x(x^2-2x-3)}{x(x^2-x-6)}$$ - Continue to factor the numerator and the denominator.

$$=\lim_{x\to 3}\frac{x(x-3)(x+1)}{x(x-3)(x+2)}$$ - Cancel common factors from the numerator and the denominator.

$$=\lim_{x\to 3}\frac{x+1}{x+2}$$ - Substitute into the limit.

$$=\frac{(3)+1}{(3)+2}$$

$$=\frac{4}{5}$$ - The answer is defined so the answer is $\frac{4}{5}$.

Alternatively, since the value of the limit is 0/0 after substitution, we can apply l'Hopital's Rule:

$$\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^3-x^2-6x}=\lim_{x\to 3}\frac{\frac{d}{dx}\left(x^3-2x^2-3x\right)}{\frac{d}{dx}\left(x^3-x^2-6x\right)}$$

- Take the derivative of the numerator and take the derivative of the denominator individually.

$$=\lim_{x\to 3}\frac{3x^2-4x-3}{3x^2-2x-6}$$

- Substitute limit into the equation.

$$=\frac{3(3)^2-4(3)-3}{3(3)^2-2(3)-6}$$

$$=\frac{27-12-3}{27-6-6}$$

- Evaluate.

$$=\frac{12}{15}$$

- Simplify.

$$=\underline{\underline{\frac{4}{5}}}$$

- The answer is defined so the answer is $\frac{4}{5}$.

A2) Find the value of $\lim_{x\to 0}\left(\frac{\sin 5x}{\sin 2x}-\frac{\sin 3x}{4x}\right)$.

$$\lim_{x\to 0}\left(\frac{\sin 5x}{\sin 2x}-\frac{\sin 3x}{4x}\right)=\frac{\sin\left[5(0)\right]}{\sin\left[2(0)\right]}-\frac{\sin\left[3(0)\right]}{4(0)}$$

- Substitute into the limit.

$$=\frac{\sin 0}{\sin 0}-\frac{\sin 0}{0}$$

- Evaluate.

$$=\frac{0}{0}-\frac{0}{0}$$

- The answer is undefined so try l'Hopital's Rule.

$$\lim_{x\to 0}\left(\frac{\sin 5x}{\sin 2x}-\frac{\sin 3x}{4x}\right)$$

- Split limit.

$$=\lim_{x\to 0}\left(\frac{\sin 5x}{\sin 2x}\right)-\lim_{x\to 0}\left(\frac{\sin 3x}{4x}\right)$$

- Take derivative of the numerator and take derivative of the denominator separately.

$$=\lim_{x\to 0}\left[\frac{\frac{d}{dx}(\sin 5x)}{\frac{d}{dx}(\sin 2x)}\right]-\lim_{x\to 0}\left[\frac{\frac{d}{dx}(\sin 3x)}{\frac{d}{dx}(4x)}\right]$$

$$=\lim_{x\to 0}\left[\frac{\overbrace{(\cos 5x)\cdot\frac{d}{dx}5x}^{\text{Chain Rule}}}{\underbrace{(\cos 2x)\cdot\frac{d}{dx}2x}_{\text{Chain Rule}}}\right]-\lim_{x\to 0}\left[\frac{\overbrace{(\cos 3x)\cdot\frac{d}{dx}3x}^{\text{Chain Rule}}}{4}\right]$$

$$=\lim_{x\to 0}\left(\frac{5\cos 5x}{2\cos 2x}\right)-\lim_{x\to 0}\left(\frac{3\cos 3x}{4}\right)$$

- Substitute into the equation.

$$=\frac{5\cos\left[5(0)\right]}{2\cos\left[2(0)\right]}-\frac{3\cos\left[3(0)\right]}{4}$$

$$=\frac{5\cos 0}{2\cos 0}-\frac{3\cos 0}{4}$$

$$=\frac{5\cdot 1}{2\cdot 1}-\frac{3\cdot 1}{4}$$

- Find common denominator.

$$=\frac{5}{2}\times\frac{2}{2}-\frac{3}{4}$$

- Evaluate.

$$=\frac{10-3}{4}$$

$$=\underline{\underline{\frac{7}{4}}}$$

- The answer is defined so the answer is $\frac{7}{4}$.

A3) Find the value of $\lim_{x\to\infty}\frac{3x^4-x^3+4}{5+x^2-8x^4}$.

$$\lim_{x\to\infty}\frac{3x^4-x^3+4}{5+x^2-8x^4}=\frac{3(\infty)^4-(\infty)^3+4}{5+(\infty)^2-8(\infty)^4}$$ - Substitute into equation.

$$=\frac{\infty}{\infty}$$ - $\frac{\infty}{\infty}$ is undefined, so try multiplying by $\frac{\frac{1}{x^n}}{\frac{1}{x^n}}$.

Multiply the expression by $\frac{\frac{1}{x^n}}{\frac{1}{x^n}}$, where n is the smaller of the largest exponent found either in the numerator or denominator, ONLY when the limit approaches $\pm\infty$.

$$\lim_{x\to\infty}\frac{3x^4-x^3+4}{5+x^2-8x^4}=\lim_{x\to\infty}\frac{\left(3x^4-x^3+4\right)}{\left(5+x^2-8x^4\right)}\times\frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$ - The maximum exponent in the numerator is 4. The maximum exponent in the denominator is 4. 4 is the smaller number of 4 and 4, so multiply limit by $\frac{1}{x^n}\Big/\frac{1}{x^n}$ with $n=4$.

$$=\lim_{x\to\infty}\frac{\frac{3x^4}{x^4}-\frac{x^3}{x^4}+\frac{4}{x^4}}{\frac{5}{x^4}+\frac{x^2}{x^4}-\frac{8x^4}{x^4}}$$

$$=\lim_{x\to\infty}\frac{3-\frac{1}{x}+\frac{4}{x^4}}{\frac{5}{x^4}+\frac{1}{x^2}-8}$$ - Substitute in limit.

$$=\frac{3-\frac{1}{\infty}+\frac{4}{\infty^4}}{\frac{5}{\infty^4}+\frac{1}{\infty^2}-8}$$ - Note: $\frac{\text{constant}}{\infty}=\frac{\text{constant}}{\infty^2}=\frac{\text{constant}}{\infty^n}=0$

$$=\frac{3-0+0}{0+0-8}$$

$$=\frac{3}{8}$$ - The answer is defined so the answer is $\frac{3}{8}$.

A4) If $f(x)=\dfrac{2x-3}{5x+4}$, then $f^{-1}(x)=$

An equivalent way of writing this expression is to substitute $f(x)$ with y.

$$f(x)=\frac{2x-3}{5x+4}$$
$$y=\frac{2x-3}{5x+4}$$

$$x=\frac{2y-3}{5y+4}$$

- Take the inverse: switch x with y and y with x.

$$(5y+4)x=2y-3$$
$$5xy+4x=2y-3$$
$$5xy-2y=-3-4x$$

- Solve for y.

$$y(5x-2)=-3-4x$$
$$y=\frac{-3-4x}{5x-2}$$
$$y=\frac{(-1)(3+4x)}{(-1)(2-5x)}$$
$$y=\frac{(3+4x)}{(2-5x)}$$

- Substitute y for $f^{-1}(x)$.

$$f^{-1}(x)=\frac{(3+4x)}{(2-5x)}$$

- The answer is $f^{-1}(x)=\dfrac{(3+4x)}{(2-5x)}$.

Or:

An equivalent way of writing this expression using the inverse function can be obtained by swapping x with $f^{-1}(x)$ from the original function, and $f(x)$ with x:

$$f(x) = \frac{2x-3}{5x+4}$$

$$x = \frac{2f^{-1}(x)-3}{5f^{-1}(x)+4}$$ - Substitute $f^{-1}(x)$ for x, and x for $f(x)$.

$$x \cdot \left(5f^{-1}(x)+4\right) = 2f^{-1}(x)-3$$

$$5xf^{-1}(x)+4x = 2f^{-1}(x)-3$$ - Solve for $f^{-1}(x)$.

$$2f^{-1}(x)-5xf^{-1}(x) = 4x+3$$

$$f^{-1}(x)\cdot(2-5x) = 4x+3$$

$$f^{-1}(x) = \frac{3+4x}{2-5x}$$ - The answer is $\frac{3+4x}{2-5x}$.

A5) If $f(x)=\sin^{-1}\left(\frac{x}{3}\right)$, then $f'\left(\sqrt{5}\right)=$

$$f(x)=\sin^{-1}\left(\frac{x}{3}\right)$$

- Change the form of the function into a form that we can differentiate: we may not know how to differentiate $f(x)=\sin^{-1}x$, but we do know how to differentiate $f(x)=\sin x$.

$$\sin\left[f(x)\right]=\frac{x}{3}$$

$$\frac{d}{dx}\left(\sin\left[f(x)\right]\right)=\frac{d}{dx}\left(\frac{x}{3}\right)$$

- Take the derivative of both sides using chain rule.

$$\underbrace{f'(x)\cdot\cos\left[f(x)\right]}_{\text{Chain Rule}}=\frac{1}{3}$$

- Solve for $f^{-1}(x)$.

$$f'(x)=\frac{1}{3\cos\left[f(x)\right]}$$

- Substitute $f(x)=\sin^{-1}(x/3)$ back into the equation. (See beginning of the question.)

$$=\frac{1}{3\cos\left[\sin^{-1}\left(\frac{x}{3}\right)\right]}$$

$$f'\left(\sqrt{5}\right)=\frac{1}{3\cos\left[\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]}$$

- Substitute $\sqrt{5}$ for *x*.

$$=\frac{1}{3\left(\frac{2}{3}\right)}$$

- $\cos\left[\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]=\frac{2}{3}$ See explanation below.

$$=\underline{\underline{\frac{1}{2}}}$$

- The answer is $\frac{1}{2}$.

To determine the value of $\cos\left[\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]$, we use the diagram below:

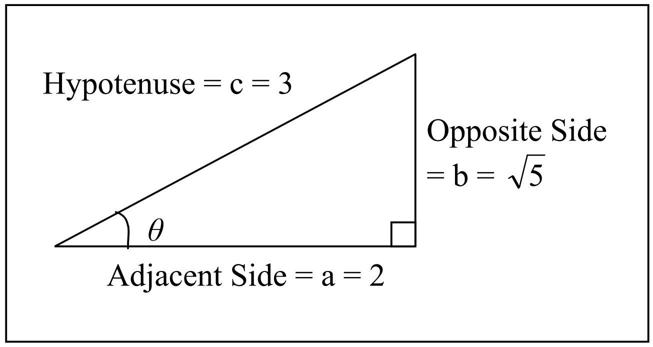
Let $\theta = \sin^{-1}\left(\frac{\sqrt{5}}{3}\right)$, then:

$$\sin\theta = \frac{\sqrt{5}}{3} = \frac{\text{Opposite Side}}{\text{Hypotenuse Side}}$$

If we draw a right-angled triangle, the side opposite θ is therefore $\sqrt{5}$ and the hypotenuse is 3.

$$\cos\left[\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right] = \cos\theta$$

-



Pythagorean Theorem states that:

$$a^2 + b^2 = c^2$$
$$a^2 + \left(\sqrt{5}\right)^2 = (3)^2$$
$$a^2 + 5 = 9$$
$$a^2 = 9 - 5$$
$$a^2 = 4$$
$$a = 2$$

$$\therefore \cos\theta = \frac{\text{Adjacent Side}}{\text{Hypotenuse Side}} = \frac{2}{3}$$

A6) If $2x^2 - xy + y^3 = 20$, find the value of $\frac{dy}{dx}$ at the point where $x = 3$, $y = 2$.

$$2x^2 - xy + y^3 = 20$$
- Differentiate both sides of the equal sign.

$$\frac{d}{dx}\left(2x^2 - xy + y^3\right) = \frac{d}{dx}(20)$$

$$\frac{d}{dx}\left(2x^2\right) - \underbrace{\frac{d}{dx}(xy)}_{\text{Product Rule}} + \underbrace{\frac{d}{dx}\left(y^3\right)}_{\text{Implicit Differentiation}} = \frac{d}{dx}(20)$$
- Use Product Rule.

$$\frac{d}{dx}\left(2x^2\right) - \underbrace{\left[\left(\frac{d}{dx}x\right)\cdot y + x\cdot\frac{d}{dx}y\right]}_{\text{Product Rule}} + \frac{d}{dx}\left(y^3\right) = \frac{d}{dx}(20)$$
- Use Chain Rule to do Implicit Differentiation.

$$2\cdot 2x - \left(y + x\cdot\frac{dy}{dx}\right) + \underbrace{3y^2\cdot\frac{d}{dx}y}_{\text{Implicit Differentiation}} = 0$$

$$x\cdot\frac{dy}{dx} - 3y^2\cdot\frac{dy}{dx} = 4x - y$$
- Solve equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx}\cdot\left(x - 3y^2\right) = 4x - y$$

$$\frac{dy}{dx} = \frac{4x - y}{x - 3y^2}$$
- The derivative is $\frac{dy}{dx} = \frac{4x - y}{x - 3y^2}$.

$$\left.\frac{dy}{dx}\right|_{x=3,\,y=2} = \frac{4\cdot 3 - 2}{3 - 3\cdot 2^2}$$
- To evaluate the derivative at the point $(3,2)$, substitute $x = 3$, and $y = 2$.

$$= \frac{12 - 2}{3 - 12}$$
- Simplify.

$$= -\frac{10}{9}$$
- The final answer is $-\frac{10}{9}$.

A7) Let $f(x) = ax^2 + bx$, where a and b are constants. If the tangent line to the curve $y = f(x)$ at the point $(1,1)$ has the equation $y = 3x - 2$, then $f(3) =$

To solve for 2 unknowns we need 2 equations. Equation 1 is from $f(1)$, while equation 2 is $f'(1)$:

$f(x) = ax^2 + bx$

$f(1) = a(1)^2 + b(1)$ - Substitute 1 for x.

$f(1) = a + b$ - Simplify.

$f(1) = 1$ - Substitute $f(1) = 1$.

$1 = a + b$ (Eqn. 1) - We now have Equation 1.

The tangent line at $(1,1)$ has the equation $y = 3x - 2$, i.e., the slope at $(1,1)$ is 3 or $f'(1) = 3$.

$f(x) = ax^2 + bx$ - Differentiate.

$f'(x) = 2ax + b$

$f'(1) = 2(1)x + b$ - Substitute 1 for x.

$f'(1) = 2x + b$

$f'(1) = 3$ - Substitute 3 for $f'(1)$.

$3 = 2x + b$ (Eqn. 2)

Or:

$\frac{d}{dx}\left(ax^2 + bx\right)\Big|_{x=1} = 3$

$2ax + b\Big|_{x=1} = 3$

$2a \cdot 1 + b = 3$ - Substitute 1 for x.

$2a + b = 3$ (Eqn. 2)

To obtain values for a and b, the system of simultaneous equations composed of Eqn's 1 and 2 need to be solved.

$a + b = 1$ (Eqn. 1)

$2a + b = 3$ (Eqn. 2)

$a = 2$ - Subtract Eqn. 1 from Eqn. 2.

$b = -1$ - Substitute $a = 2$ into Eqn. 1.

Therefore, the function $f(x)$ is:

$$
\begin{aligned}
f(x) &= 2x^2 - x \\
f(3) &= 2 \cdot 3^2 - 3 \\
&= 18 - 3 \\
&= \underline{\underline{15}}
\end{aligned}
$$

A8) If m is a fixed constant such that the two families of curves $y=\frac{1}{x+c}$ and $y=m\left(x+k\right)^{\frac{1}{3}}$ are orthogonal trajectories of each other, what must be the value of m?

Let $f(x)=\frac{1}{x+c}$ and $g(x)=m(x+k)^{\frac{1}{3}}$ then:

$$f'(x)\cdot g'(x)=-1$$ - Definition of orthogonal trajectories.

$$\frac{d}{dx}\left(\frac{1}{x+c}\right)\cdot\frac{d}{dx}\left[m\left(x+k\right)^{\frac{1}{3}}\right]=-1$$ - Evaluate the derivatives.

$$\frac{-1}{\left(x+c\right)^2}\cdot\frac{1}{3}m\left(x+k\right)^{-\frac{2}{3}}=-1 \qquad \text{(Eqn. 1)}$$ - This is 1 of the 2 simultaneous equations needed.

$$f\left(x\right)=g\left(x\right)$$ - Now we find where the 2 curves intersect.

$$\frac{1}{x+c}=m\left(x+k\right)^{\frac{1}{3}} \qquad \text{(Eqn. 2)}$$ - This is the 2nd equation needed. We rewrite it so that we can substitute this into Eqn. 1.

$$\left(\frac{1}{x+c}\right)^2=\left[m\left(x+k\right)^{\frac{2}{3}}\right]^2$$ - First we square both sides.

$$\frac{1}{\left(x+c\right)^2}=m^2\left(x+k\right)^{\frac{2}{3}} \qquad \text{(Eqn. 2')}$$ - We then simplify to give the first term in Eqn. 1. We can then substituting Eqn. 2’ into Eqn. 1.

$$-m^2\left(x+k\right)^{\frac{2}{3}}\cdot\frac{1}{3}m\left(x+k\right)^{-\frac{2}{3}}=-1$$ - We can simplify this expression by first multiplying both sides by -3.

$$m^2\left(x+k\right)^{\frac{2}{3}}\cdot m\left(x+k\right)^{-\frac{2}{3}}=3$$ - The $\left(x+k\right)$ terms cancel out.

$$m^3=3$$

$$m=\underline{\underline{\sqrt[3]{3}}}$$

A9) Let $h(x)=\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials. Consider the following three statements.

I. If $\lim_{x\to 5} g(x)=g(5)$ and h is continues at 5, then f must be continuous at 5.

II. If $(x-4)$ is a factor of $g(x)$, then the line $x=4$ must be a vertical asymptote of the curve $y=h(x)$.

III. If $g(3)<0$ and $\dfrac{g(3)}{g'(3)}=\dfrac{f(3)}{f'(3)}$, then the curve $y=h(x)$ must have a horizontal tangent line at $x=3$.

I: **TRUE**

Since $\lim_{x\to 5} g(x)=g(5)$, $g(x)$ is continuous at $x=5$. Since $h(x)=\dfrac{f(x)}{g(x)}$, $f(x)=g(x)\cdot h(x)$ must therefore also be continuous at $x=5$ given that both $g(x)$ and $h(x)$ are continuous here.

II: **FALSE**

$(x-4)$ being a factor of $g(x)$ means that $g(4)=0$. This yields a vertical asymptote at $x=4$, i.e. that $h(4)=\dfrac{f(4)}{g(4)}=\pm\infty$, only if $f(4)\neq 0$, otherwise $\lim_{x\to 4}\dfrac{f(x)}{g(x)}$ may have a definite value.

III: **TRUE**

To show that there is a horizontal tangent at $x=3$, we need to show that $h'(3)=0$.

$$h'(3)=\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]\Bigg|_{x=3}$$

$$=\underbrace{\frac{f'(3)\cdot g(3)-f(3)\cdot g'(3)}{\left[g(3)\right]^2}}_{\text{Quotient Rule}} \qquad \text{(Eqn. 1)}$$

- To simplify Eqn. 1 further, we make use of the following given condition.

$$\frac{g(3)}{g'(3)}=\frac{f(3)}{f'(3)}$$

$$g(3)=\frac{f(3)\cdot g'(3)}{f'(3)} \qquad \text{(Eqn. 2)}$$

- We can now substituting Eqn. 2 into Eqn. 1 to replace the $g(3)$ in the numerator.

$$h'(3) = \frac{f'(3)\cdot\frac{f(3)\cdot g'(3)}{f'(3)} - f(3)\cdot g'(3)}{\left[g(3)\right]^2}$$

- Note that the $f'(3)$ cancel out.

$$= \frac{f(3)\cdot g'(3) - f(3)\cdot g'(3)}{\left[g(3)\right]^2}$$

- There are 2 of the same term in the numerator.

$$= \frac{0}{\left[g(3)\right]^2}$$

- This is either 0, or undefined if $g(3) = 0$.

$$= 0$$

- Since $g(3) < 0$, i.e. $g(3) \neq 0$.

A10) If $y^2 + 2e^{-xy} = 6$, find the value of $\dfrac{d^2y}{dx^2}$ at the point where $y = 2$.

$$2^2 + 2e^{-2x} = 6$$
$$e^{-2x} = 1$$
$$-2x = \ln 1$$
$$-2x = 0$$
$$x = 0$$

- First determine value of x corresponding to $y = 2$.
- Thus, we want to find the 2nd derivative at $(0,2)$.

$$\frac{d}{dx}\left(y^2 + 2e^{-xy}\right) = \frac{d}{dx}(6)$$

- We next find the 1st derivative. We need both the Chain Rule and Product Rule here.

$$\underbrace{2y \cdot \frac{d}{dx}y}_{\text{Chain Rule}} + \underbrace{2e^{-xy} \cdot \frac{d}{dx}(-xy)}_{\text{Chain Rule}} = 0$$

- Product Rule is needed here.

$$2y\frac{dy}{dx} + 2e^{-xy}\underbrace{\left\{\left[\frac{d}{dx}(-x)\right]\cdot y + (-x)\cdot\frac{d}{dx}y\right\}}_{\text{Product Rule}} = 0$$

$$2y\frac{dy}{dx} + 2e^{-xy}\left(-y - x\cdot\frac{dy}{dx}\right) = 0$$

- Now we remove the brackets.

$$2y\frac{dy}{dx} - 2ye^{-xy} - 2xe^{-xy}\frac{dy}{dx} = 0$$

- Solve for $\dfrac{dy}{dx}$.

$$2y\frac{dy}{dx} - 2xe^{-xy}\frac{dy}{dx} = 2ye^{-xy}$$
$$\left(2y - 2xe^{-xy}\right)\frac{dy}{dx} = 2ye^{-xy}$$
$$\frac{dy}{dx} = \frac{ye^{-xy}}{y - xe^{-xy}}$$

- We can now evaluate the derivative at $(0,2)$. Let's first evaluate the term e^{-xy} at this point.

$$e^{-xy}\Big|_{x=0,y=2} = e^{-0\cdot 2}$$
$$= 1 \qquad \text{(Eqn. 1)}$$

- We can now use this to evaluate the derivative.

$$\frac{dy}{dx}\bigg|_{x=0,y=2} = \frac{2\cdot 1}{2 - 0\cdot 1}$$
$$= 1 \qquad \text{(Eqn. 2)}$$

- Substitute $e^{-xy}\Big|_{x=0,y=2} = 1$ into derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{ye^{-xy}}{y - xe^{-xy}}\right)$$

- We now find the 2nd derivative by using the Quotient Rule.

$$= \frac{\left(\frac{d}{dx}ye^{-xy}\right)\cdot\left(y - xe^{-xy}\right) - ye^{-xy}\cdot\frac{d}{dx}\left(y - xe^{-xy}\right)}{\left(y - xe^{-xy}\right)^2}$$

- Let's now evaluate each term in turn, at the point $(0,2)$.

$$\frac{d}{dx}\left(ye^{-xy}\right)$$

- Product Rule is needed here.

$$=\underbrace{\left(\frac{d}{dx}y\right)\cdot e^{-xy}+y\cdot\frac{d}{dx}\left(e^{-xy}\right)}_{\text{Product Rule}}$$

- Chain Rule is needed here.

$$=\frac{dy}{dx}\cdot e^{-xy}+y\underbrace{\left[e^{-xy}\cdot\frac{d}{dx}(-xy)\right]}_{\text{Chain Rule}}$$

- A further application of the Product Rule is required.

$$=e^{-xy}\frac{dy}{dx}+ye^{-xy}\underbrace{\left\{\left[\frac{d}{dx}(-x)\right]\cdot y+(-x)\cdot\frac{d}{dx}y\right\}}_{\text{Product Rule}}$$

$$=e^{-xy}\frac{dy}{dx}+ye^{-xy}\left(-y-x\frac{dy}{dx}\right)$$

- Now we evaluate this at $(0,2)$. Recall from Eqn. 1 that $e^{-xy}\Big|_{x=0,y=2}=1$ and from Eqn. 2 that $\frac{dy}{dx}\Big|_{x=0,y=2}=1$.

$$\begin{aligned}\frac{d}{dx}\left(ye^{-xy}\right)\Big|_{x=0,y=2}&=1\cdot 1+2\cdot 1\cdot(-2-0\cdot 1)\\&=1+2\cdot(-2)\\&=-3\end{aligned}$$

$$\begin{aligned}\left(y-xe^{-xy}\right)\Big|_{x=0,y=2}&=2-0\cdot 1\\&=2\end{aligned}$$

- Now we evaluate the next term at $(0,2)$.

$$\begin{aligned}ye^{-xy}\Big|_{x=0,y=2}&=2\cdot 1\\&=2\end{aligned}$$

- And the next term.

$$\frac{d}{dx}\left(y - xe^{-xy}\right)$$

- And finally the last term. The Product Rule is needed here.

$$= \frac{dy}{dx} - \underbrace{\left[\left(\frac{d}{dx}x\right)\cdot e^{-xy} + x\cdot\frac{d}{dx}\left(e^{-xy}\right)\right]}_{\text{Product Rule}}$$

- Followed by the Chain Rule.

$$= \frac{dy}{dx} - \left\{e^{-xy} + x\underbrace{\left[e^{-xy}\cdot\frac{d}{dx}(-xy)\right]}_{\text{Chain Rule}}\right\}$$

- And the Product Rule once more.

$$= \frac{dy}{dx} - \left\{e^{-xy} + xe^{-xy}\underbrace{\left\{\left[\frac{d}{dx}(-x)\right]\cdot y + (-x)\cdot\frac{d}{dx}y\right\}}_{\text{Product Rule}}\right\}$$

$$= \frac{dy}{dx} - \left[e^{-xy} + xe^{-xy}\left(-y - x\frac{dy}{dx}\right)\right]$$

- Now we evaluate this at $(0,2)$. Recall from Eqn. 1 that $e^{-xy}\Big|_{x=0,y=2} = 1$ and from Eqn. 2 that $\frac{dy}{dx}\Big|_{x=0,y=2} = 1$.

$$\left.\frac{d^2y}{dx^2}\right|_{x=0,y=2} = \frac{-3\cdot 2 - 2\cdot 0}{2^2}$$

- Finally, we substitute all the terms back in.

$$= \frac{-6}{4}$$

$$= -\frac{3}{2}$$

- And arrive at the answer.

B1) Let $f(x)=\sqrt{x}$. Find $f'(x)$ from first principles (i.e. by using <u>only the definition of the derivative</u>).

$f'(x)=\lim_{a\to x}\frac{f(a)-f(x)}{a-x}$	- Definition of derivative.
$=\lim_{a\to x}\frac{\sqrt{a}-\sqrt{x}}{a-x}$	- Substitute in our function $f(x)=\sqrt{x}$.
$=\lim_{a\to x}\frac{(\sqrt{a}-\sqrt{x})\cdot(\sqrt{a}+\sqrt{x})}{(a-x)\cdot(\sqrt{a}+\sqrt{x})}$	- Since we know the derivative should have only a $\sqrt{x}$ term in the denominator. We try removing the root terms in the numerator.
$=\lim_{a\to x}\frac{(\sqrt{a})^2-(\sqrt{x})^2}{(a-x)\cdot(\sqrt{a}+\sqrt{x})}$	- Difference of squares in the numerator.
$=\lim_{a\to x}\frac{a-x}{(a-x)\cdot(\sqrt{a}+\sqrt{x})}$	- $(a-x)$ is a common factor top and bottom.
$=\lim_{a\to x}\frac{1}{\sqrt{a}+\sqrt{x}}$	- Now we can sub in the limit.
$=\frac{1}{\sqrt{x}+\sqrt{x}}$	
$=\underline{\underline{\frac{1}{2\sqrt{x}}}}$	- The answer, as should be.

B2) Use any suitable method to find $\frac{dy}{dx}$ for each of the following. There is no need to simplify your final answers for the question.

a) $y = x^2 e^{3x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 e^{3x}\right)$$

$$= \underbrace{\left(\frac{d}{dx}x^2\right)\cdot e^{3x} + x^2 \cdot \frac{d}{dx}e^{3x}}_{\text{Product Rule}}$$

$$= 2xe^{3x} + x^2 \underbrace{\left[e^{3x} \cdot \frac{d}{dx}(3x)\right]}_{\text{Chain Rule}}$$

$$= \underline{\underline{2xe^{3x} + 3x^2 e^{3x}}}$$

b) $y = \frac{2+x^2}{(3+x)^5}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{2+x^2}{(3+x)^5}\right]$$

$$= \underbrace{\frac{\left[\frac{d}{dx}\left(2+x^2\right)\right]\cdot(3+x)^5 - \left(2+x^2\right)\cdot\frac{d}{dx}(3+x)^5}{\left[(3+x)^5\right]^2}}_{\text{Quotient Rule}}$$

$$= \frac{2x(3+x)^5 - \left(2+x^2\right)\overbrace{\left[5(3+x)^4 \cdot \frac{d}{dx}(3+x)\right]}^{\text{Chain Rule}}}{(3+x)^{10}}$$

$$= \frac{2x(3+x)^5 - \left(2+x^2\right)\left[5(3+x)^4 \cdot 1\right]}{(3+x)^{10}}$$

$$= \underline{\underline{\frac{2x(3+x)^5 - 5\left(2+x^2\right)(3+x)^4}{(3+x)^{10}}}}$$

c) $y = \tan\left(\sqrt{x}\right)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\tan\left(\sqrt{x}\right)\\ &= \underbrace{\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)}_{\text{Chain Rule}}\\ &= \underline{\underline{\frac{\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}}}\end{aligned}$$

d) $y = 5^{\tan^{-1}\left(x^2\right)}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[5^{\tan^{-1}\left(x^2\right)}\right]\\ &= \underbrace{\left(5^{\tan^{-1}\left(x^2\right)}\ln 5\right)\cdot\frac{d}{dx}\left[\tan^{-1}\left(x^2\right)\right]}_{\text{Chain Rule}}\\ &= \left(5^{\tan^{-1}\left(x^2\right)}\ln 5\right)\cdot\underbrace{\frac{1}{1+\left(x^2\right)^2}\cdot\frac{d}{dx}\left(x^2\right)}_{\text{Chain Rule}}\\ &= \underline{\underline{\left(5^{\tan^{-1}\left(x^2\right)}\ln 5\right)\cdot\frac{2x}{1+x^4}}}\end{aligned}$$

B3) Find the line which passes through the point $\left(0,\frac{1}{4}\right)$ and is tangent to the curve $y=x^3$ at some point.

Let $f(x)=x^3$, and let the point (a,b) be on the curve $y=f(x)$ such that the line tangent to this point passes through the point $\left(0,\frac{1}{4}\right)$.

$$\begin{aligned}(a,b)&=(a,f(a))\\&=(a,a^3)\end{aligned}$$

$$\begin{aligned}slope&=f'(a)\\&=\frac{d}{dx}x^3\Big|_{x=a}\\&=3x^2\Big|_{x=a}\\&=3a^2\end{aligned}$$

- Now we write an expression for the slope of the line we seek.

$$\frac{a^3-\frac{1}{4}}{a-0}=3a^2$$

- We can determine what *a* is by writing an expression for the slope between the 2 points and solve for *a*.

$$a^3-\frac{1}{4}=3a^3$$

- Simplify

$$2a^3=-\frac{1}{4}$$

$$a=\sqrt[3]{-\frac{1}{8}}$$

$$a=-\frac{1}{2}$$

$$\begin{aligned}m&=3a^2\\&=3\cdot\left(-\frac{1}{2}\right)^2\\&=\frac{3}{4}\end{aligned}$$

- Now we calculate the slope *m* of the line $y=mx+c$.

The y-intercept is $\frac{1}{4}$ since we are told that the point $\left(0,\frac{1}{4}\right)$ is on the line. Therefore, our line is:

$$\underline{\underline{y=\frac{3}{4}x+\frac{1}{4}}}$$

B4) Let $f(x)=\begin{cases} 2x^3 & \text{if } x<2 \\ x^2+ax+b & \text{if } -2<x<2 \\ 3x^4-48 & \text{if } x\geq 2 \end{cases}$. Find the values of the constants a and b so that f is continuous everywhere.

Condition for continuity:

$$\left(2x^3+16\right)\Big|_{x=-2} = \Big|\left(x^2+ax+b\right)\Big|_{x=-2} \qquad (\text{Eqn. }1)$$

$$\left(x^2+ax+b\right)\Big|_{x=2} = \left(3x^4-48\right)\Big|_{x=2} \qquad (\text{Eqn. }2)$$

$2\cdot(-2)^3+16=(-2)^2+a\cdot(-2)+b$ - Simplify Eqn. 1 by putting in $x=-2$.

$-16+16=4-2a+b$

$2a-b=4$ (Eqn. 1')

$2^2+a\cdot 2+b=3\cdot 2^4-48$ - Simplify Eqn. 2 by putting in $x=2$.

$4+2a+b=48-48$

$2a+b=-4$ (Eqn. 2') - We can now solve the simultaneous eqns 1 and 2.

$(2a-b)+(2a+b)=4-4$ - Add Eqn. 1’ to Eqn. 2’.

$4a=0$

$a=\underline{\underline{0}}$ - This is one of our answers.

$2\cdot 0-b=4$ - Substitute $a=0$ into Eqn 1’.

$b=\underline{\underline{-4}}$ - This is the other answer.

B5) An object is being thrown upward so that its height (above ground) t seconds after it is thrown is $100+48t-16t^2$ feet. Find the velocity of the object when it is 132 feet above ground on its way up.

Let the height above ground be $h(t)=100+48t-16t^2$. The velocity is then the derivative $h'(t)$:

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left(100+48t-16t^2\right) \\ &= 48-32t \end{aligned}$$

$h(t)=132$	- We need to first determine the time corresponding to the height of 132 feet.
$100+48t+16t^2=132$	- Substitute in function.
$16t^2+48t-32=0$	- Subtract both sides by 132.
$t^2+3t-2=0$	- Divide both sides by 16.
$(t-1)(t-2)=0$	
$t=1$	- Ignore later time ($t=2$) since that is when the object falls back down instead.

$$\begin{aligned} h'(1) &= 48-32\cdot 1^2 \\ &= \underline{\underline{16\ (\text{feet per second})}} \end{aligned}$$

B6) Evaluate $\lim_{x\to\infty}\left(\sqrt[3]{x^3+5x^2}-x\right)$. Remember to fully justify your answer.

$$\lim_{x\to\infty}\left(\sqrt[3]{x^3+5x^2}-x\right)$$

- First try substitute in the limit.

$$= \sqrt[3]{\infty^3+5\cdot\infty^2}-\infty$$
$$= \infty-\infty$$

- This limit is undefined.

$$\lim_{x\to\infty}\left(\sqrt[3]{x^3+5x^2}-x\right)$$

- One of the problems we're faced with is the cube-root term. Let's try to simplify that.

$$= \lim_{x\to\infty}\left(\sqrt[3]{x^3\cdot\frac{\left(x^3+5x^2\right)}{x^3}}-x\right)$$

- Multiply x^3/x^3 inside the cube root.

$$= \lim_{x\to\infty}\left(\sqrt[3]{x^3}\cdot\sqrt[3]{\frac{x^3}{x^3}+\frac{5x^2}{x^3}}-x\right)$$

- Distribute the cube root.

$$= \lim_{x\to\infty}\left(x\cdot\sqrt[3]{1+\frac{5}{x}}-x\right)$$

$$= \lim_{h\to0}\left(\frac{5}{h}\cdot\sqrt[3]{1+h}-\frac{5}{h}\right)$$

- $\lim_{x\to\infty}\frac{5}{x}=0$, so we can make the substitution $h=\frac{5}{x}$ (and therefore $x=\frac{5}{h}$) if we change the limit from $x\to\infty$ to $h\to0^+$. (Limit is from positive since the original limit is to positive infinity.)

$$= 5\cdot\lim_{h\to0}\left(\frac{\sqrt[3]{1+h}-1}{h}\right) \qquad (\text{Eqn. } 1)$$

- Group into one fraction.

This expression bares resemblance to the definition of a derivative, which states that:

$$f'(a)=\lim_{h\to0}\left(\frac{f(x+a)-f(a)}{h}\right)$$

This is the same as our limit if $f(x)=\sqrt[3]{x}$ and $a=1$:

$$\lim_{h\to0}\left(\frac{f(x+a)-f(a)}{h}\right)=\lim_{h\to0}\left(\frac{\sqrt{x+1}-\sqrt{1}}{h}\right)$$
$$=\lim_{h\to0}\left(\frac{\sqrt{1+x}-1}{h}\right)$$

The only apparent difference left between this limit and our limit in Eqn. 1 is that there is an x in the numerator here, while an h is present in Eqn. 1. However, it is irrelevant what symbol we use to designate the variable of a function, such that we can write $f(h)=\sqrt[3]{h}$ instead of $f(x)=\sqrt[3]{x}$.

$$\lim_{x\to\infty}\left(\sqrt[3]{x^3+5x^2}-x\right)$$

$$= 5\cdot\lim_{h\to 0}\left(\frac{\sqrt[3]{1+h}-1}{h}\right)$$

- This limit can therefore be written as a derivative.

$$= 5\cdot f'(1)$$

$$= 5\cdot\frac{d}{dx}\left(\sqrt[3]{x}\right)\bigg|_{x=1}$$

- We now take the derivative.

$$= 5\cdot\frac{1}{3}x^{-\frac{2}{3}}\bigg|_{x=1}$$

- And substitute in $x=1$.

$$= 5\cdot\left(\frac{1}{3}\cdot 1^{-\frac{2}{3}}\right)$$

$$= \underline{\underline{\frac{5}{3}}}$$

- This is the answer.