A3) If $f(x)=\dfrac{3x+1}{2x-5}$, find the expression for $f^{-1}(x)$. [Chapter 1]

A5) Find the value of $\lim\limits_{x\to 0}\dfrac{\sin(2x)\cos(3x)}{\sin(5x)\cos(4x)}$. [Chapter 3]

A6) Evaluate $\lim\limits_{x\to 2}\dfrac{\sqrt{x+2}-2}{x-2}$. [Chapter 3]

A7) The graph $y=e^{3x^3-3x^2+x-4}$ has a horizontal tangent line at what value of x? [Chapter 3]

A8) A spaceship approaching touchdown on the plant GRY has height $y=100-100t+25t^2$ meters at t seconds. What is its speed in meters per second when it hits the ground? [Chapter 3]

A10) Let $f(x)=\begin{cases}-x^2 & \text{if } x\le 0\\ 8\sqrt{x} & \text{if } x>0\end{cases}$. There is one line that is tangent to the graph of f at two points. What is its slope? [Chapter 2]

B1) Find $f'(x)$ from first principles (i.e. by using only the definition of the derivative) for $f(x)=x^2$. [Chapter 2]

B2) Find any suitable method to find $\dfrac{dy}{dx}$ for each of the following: [Chapter 3]

a) $y=(2+x^3)e^{2x}$

b) $y=\dfrac{\cos x}{1+x^4}$

c) $y=\sin^{-1}x^2$

d) $y=5^{\tan 2x}$

B3) Let $f(x)=\begin{cases}cx+1 & \text{if } x<3\\ cx^2-1 & \text{if } x\ge 3\end{cases}$. Find the value of c so that f is continuous everywhere. [Chapter 2]

B4) Find the non-horizontal line passing through the point $\left(\dfrac{3}{4},0\right)$ and tangent to the curve $y=x^2$ at some point. [Chapter 2]

B5) Find $\dfrac{dy}{dx}$ if $y^3+xy-3y^2-2x-2=0$. What is the value of $\dfrac{dy}{dx}$ when $x=2$? [Chapter 3]

B6) Suppose that f is a differentiable function such that $f'(8)=2$. Find the value of $\lim\limits_{x\to 8}\dfrac{f(x)-f(8)}{x^{\frac{1}{3}}-2}$. [Chapter 3]

A3) If $f(x)=\dfrac{3x+1}{2x-5}$, find the expression for $f^{-1}(x)$. [Chapter 1]

$$x=\frac{3y+1}{2y-5}$$ - To find the inverse function, we swap $f(x)$ for *x* and *x* for *y*, and then solve for *y*.

$$x(2y-5)=3y+1$$
$$2xy-5x=3y+1$$
$$2xy-3y=1+5x$$
$$(2x-3)y=1+2x$$
$$y=\underline{\underline{\frac{5x+1}{2x-3}}}$$

A5) Find the value of $\lim_{x\to 0}\frac{\sin(2x)\cos(3x)}{\sin(5x)\cos(4x)}$. [Chapter 3]

$$\lim_{x\to 0}\frac{\sin(2x)\cos(3x)}{\cos(5x)\cos(4x)}$$

- We first write this fraction into a product of convenient fractions.

$$=\lim_{x\to 0}\left[\frac{\sin(2x)}{\sin(5x)}\cdot\frac{\cos(3x)}{\cos(4x)}\right]$$

- We can now distribute the limit to get 2 individual limits to evaluate.

$$=\lim_{x\to 0}\frac{\sin(2x)}{\sin(5x)}\cdot\lim_{x\to 0}\frac{\cos(3x)}{\cos(4x)}$$

- The 1st limit is in the form $0/0$, so we apply l'Hopital's Rule: take the derivative top and bottom separately.

$$=\lim_{x\to 0}\frac{\frac{d}{dx}\sin(2x)}{\frac{d}{dx}\sin(5x)}\cdot\lim_{x\to 0}\frac{\cos(3x)}{\cos(4x)}$$

- The Chain Rule is required for each of the derivatives.

$$=\lim_{x\to 0}\frac{\cos(2x)\frac{d}{dx}2x}{\cos(5x)\frac{d}{dx}5x}\cdot\lim_{x\to 0}\frac{\cos(3x)}{\cos(4x)}$$

$$=\lim_{x\to 0}\frac{2\cos(2x)}{5\cos(5x)}\cdot\lim_{x\to 0}\frac{\cos(3x)}{\cos(4x)}$$

- Now we can substitute in $x=0$ to evaluate the value of the limit.

$$=\frac{2\cos(2\cdot 0)}{5\cos(5\cdot 0)}\cdot\frac{\cos(3\cdot 0)}{\cos(4\cdot 0)}$$

$$=\frac{2\cdot 1}{5\cdot 1}\cdot\frac{1}{1}\cdot$$

$$=\underline{\underline{\frac{2}{5}}}$$

A6) Evaluate $\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}$. [Chapter 3]

$$\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}$$

- We first multiply top and bottom by $\sqrt{x+2}+2$ to yield a difference of squares on the numerator..

$$=\lim_{x\to 2}\left(\frac{\sqrt{x+2}-2}{x-2}\cdot\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}\right)$$

$$=\lim_{x\to 2}\frac{\left(\sqrt{x+2}\right)^2-2^2}{(x-2)\left(\sqrt{x+2}+2\right)}$$

$$=\lim_{x\to 2}\frac{x+2-4}{(x-2)\left(\sqrt{x+2}+2\right)}$$

$$=\lim_{x\to 2}\frac{x-2}{(x-2)\left(\sqrt{x+2}+2\right)}$$

- We can now cancel out the common factor $x-2$.

$$=\lim_{x\to 2}\frac{1}{\sqrt{x+2}+2}$$

- Apply the limit.

$$=\frac{1}{\sqrt{2+2}+2}$$

$$=\underline{\underline{\frac{1}{4}}}$$

A7) The graph $y = e^{3x^3 - 3x^2 + x - 4}$ has a horizontal tangent line at what value of x? [Chapter 3]

$$\frac{d}{dx} e^{3x^3-3x^2+x-4} = 0$$
- A horizontal tangent line means that the derivative is 0, so we take the derivative of the function, using the Chain Rule.

$$e^{3x^3-3x^2+x-4} \frac{d}{dx}\left(3x^3 - 3x^2 + x - 4\right) = 0$$

$$e^{3x^3-3x^2+x-4}\left(9x^2 - 6x^2 + 1\right) = 0$$
- Factorize the quadratic.

$$e^{3x^3-3x^2+x-4}\left(3x-1\right)^2 = 0$$
- An exponent of ∞ is needed to make the 1st term 0, so it is ignored.

$$3x - 1 = 0$$
- Solve for x.

$$x = \underline{\underline{\frac{1}{3}}}$$

A8) A spaceship approaching touchdown on the plant GRY has height $y = 100 - 100t + 25t^2$ meters at t seconds. What is its speed in meters per second when it hits the ground? [Chapter 3]

$$\begin{aligned} v &= \frac{d}{dt}\left(100 - 100t + 25t^2\right) \\ &= -100 + 50t \end{aligned}$$ - Speed is the derivative of distance with respect to time.

$$\begin{aligned} 100 - 100t + 25t^2 &= 0 \\ t^2 - 4t + 4 &= 0 \\ (t-2)^2 &= 0 \\ t &= 2 \end{aligned}$$ - We next find the time corresponding to spaceship at 0 meter above ground.

$$\begin{aligned} v(2) &= -100 + 50(2) \\ &= -100 + 100 \\ &= \underline{\underline{0 \text{ (meter per second)}}} \end{aligned}$$ - Substitute $t = 2$ into the expression for speed.

A10) Let $f(x)=\begin{cases}-x^2 & \text{if } x\le 0\\ 8\sqrt{x} & \text{if } x>0\end{cases}$. There is one line that is tangent to the graph of *f* at two points. What is its slope?

[Chapter 2]

$\frac{d}{dx}\left(-x^2\right)=-2x$ - Let's evaluate the expression for the slope of *f* when *x* is negative. We'll call this Equation 1.

$\frac{d}{dx}8\sqrt{x}=\frac{8}{2\sqrt{x}}$ - Let's also evaluate the expression for the slope of *f* when *x* is positive.

$=\frac{4}{\sqrt{x}}$ - We'll call this Equation 2.

Let the point at negative *x* that the line is tangent to be $\left(a,-a^2\right)$, and the corresponding point at positive *x* be $\left(b,8\sqrt{b}\right)$.

$\left.\frac{d}{dx}\left(-x^2\right)\right|_{x=a}=\left.\frac{d}{dx}8\sqrt{x}\right|_{x=b}$ - The slopes evaluated at the 2 points must be the same, since both of these points are on the tangent line. Thus we set the expressions from Equations 1 and 2 to be equal.

$-2a=\frac{4}{\sqrt{b}}$

$\sqrt{b}=-\frac{2}{a}$ - Let's call this Equation 3.

$b=\left(-\frac{2}{a}\right)^2$

$=\frac{4}{a^2}$ - Let's call this Equation 4.

$\frac{8\sqrt{b}-\left(-a^2\right)}{b-a}=\left.\frac{d}{dx}\left(-x^2\right)\right|_{x=a}$ - The slope calculated from the coordinates of the 2 points must also be the same as the slope be the same as that calculated from taking the derivative. We'll use the expression from Equation 1 here.

$\frac{8\sqrt{b}+a^2}{b-a}=-2a$ - Let's substitute the results from Equations 1 and 2 to replace all occurrences of *b* with *a*.

$\frac{8\left(-\frac{2}{a}\right)+a^2}{\left(-\frac{2}{a}\right)^2-a}=-2a$

$\frac{-\frac{16}{a}+a^2}{\frac{4}{a^2}-a}=-2a$ - We can simplify this by multiplying the left side top and bottom by a^2.

$\frac{-\frac{16}{a}+a^2}{\frac{4}{a^2}-a}\cdot\frac{a^2}{a^2}=-2a$

$$\frac{-16a + a^4}{4 - a^3} = -2a$$

$$-16a + a^4 = -2a\left(4 - a^3\right)$$

$$-16a + a^4 = -8a + 2a^4$$ - Move all terms to once side and factorize.

$$a^4 + 8a = 0$$

$$a\left(a^3 + 8\right) = 0$$

$$a\left(a + 2\right)\left(a^2 - 2a + 4\right) = 0$$

There are therefore 3 possibilities:

$a = 0$ - The function is not differentiable here since the function is not smooth at this point. This root therefore ignored.

$a = -2$ - This is a valid root.

$a^2 - 2a + 4 = 0$ - Solve the quadratic using the quadratic equation.

$$a = \frac{-\left(-2\right) \pm \sqrt{\left(-2\right)^2 - 4\left(1\right)\left(4\right)}}{2\left(1\right)}$$

$$= \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm \sqrt{-14}}{2}$$ - These roots are imaginary, and so are ignored.

$$\left.\frac{d}{dx}\left(-x^2\right)\right|_{x=-2} = \left.-2x\right|_{x=-2}$$ - Finally we can evaluate the value of the slope by using Equation 1.

$$= -2\left(-2\right)$$

$$= \underline{\underline{-4}}$$

B1) Find $f'(x)$ from first principles (i.e. by using only the definition of the derivative) for $f(x) = x^2$. [Chapter 2]

$f'(x) = \lim_{h \to x} \frac{f(h) - f(x)}{h - x}$ - This is one way to write the definition of the derivative. Let's substitute in the function.

$= \lim_{h \to x} \frac{h^2 - x^2}{h - x}$ - We can simplify this by factorizing the difference of squares on the numerator.

$= \lim_{h \to x} \frac{(h - x)(h + x)}{h - x}$ - Cancel out the common factor.

$= \lim_{h \to x} (h + x)$ - We can now apply the limit.

$= x + x$

$= \underline{\underline{2x}}$

B2) Find any suitable method to find $\frac{dy}{dx}$ for each of the following: [Chapter 3]

a) $y=\left(2+x^3\right)e^{2x}$

b) $y=\frac{\cos x}{1+x^4}$

c) $y=\sin^{-1}x^2$

d) $y=5^{\tan 2x}$

a)

$$\frac{dy}{dx}=\frac{d}{dx}\left[\left(2+x^3\right)e^{2x}\right]$$ - The Product Rule is needed here.

$$=\left[\frac{d}{dx}\left(2+x^3\right)\right]e^{2x}+\left(2+x^3\right)\frac{d}{dx}e^{2x}$$ - The Chain Rule is needed for the derivative on the left.

$$=3x^2e^{2x}+\left(2+x^3\right)e^{2x}\frac{d}{dx}2x$$

$$=\underline{\underline{3x^2e^{2x}+2\left(2+x^3\right)e^{2x}}}$$

b)

$$\frac{dy}{dx}=\frac{d}{dx}\frac{\cos x}{1+x^4}$$ - The Quotient Rule is needed here.

$$=\frac{\left(\frac{d}{dx}\cos x\right)\left(1+x^4\right)-\cos x\frac{d}{dx}\left(1+x^4\right)}{\left(1+x^4\right)^2}$$

$$=\underline{\underline{\frac{-\left(1+x^4\right)\sin x-4x^3\cos x}{\left(1+x^4\right)^2}}}$$

c)

$$\frac{dy}{dx}=\frac{d}{dx}\sin^{-1}x^2$$ - The Chain Rule is needed here.

$$=\frac{1}{\sqrt{1-\left(x^2\right)^2}}\frac{d}{dx}x^2$$

$$=\underline{\underline{\frac{2x}{\sqrt{1-x^4}}}}$$

d)

$$\frac{dy}{dx} = \frac{d}{dx}5^{\tan 2x}$$ - The Chain Rule is needed here.

$$= \left(5^{\tan 2x}\ln 5\right)\frac{d}{dx}\tan 2x$$ - The Chain Rule is needed again here.

$$= \left(5^{\tan 2x}\ln 5\right)\sec^2 2x\frac{d}{dx}2x$$

$$= \underline{\underline{2\left(5^{\tan 2x}\ln 5\right)\sec^2 2x}}$$

B3) Let $f(x)=\begin{cases} cx+1 & \text{if } x<3 \\ cx^2-1 & \text{if } x\geq 3 \end{cases}$. Find the value of *c* so that *f* is continuous everywhere. [Chapter 2]

$$\lim_{x\to 3^-}(cx+1)=\lim_{x\to 3^+}(cx^2-1)$$

- Since both $cx+1$ and cx^2-1 are continuous, the only place we need to make sure is continuous is when $x=3$.

$$3c+1=3^2c-1$$

$$3c+1=9c-1$$

- Solve for *c*.

$$6c=2$$

$$c=\frac{1}{3}$$

B4) Find the non-horizontal line passing through the point $\left(\frac{3}{4},0\right)$ and tangent to the curve $y=x^2$ at some point.

[Chapter 2]

$$\frac{dy}{dx}=\frac{d}{dx}x^2 = 2x$$
- Let's first evaluate the derivative.

Let the point on the curve that the line is tangent to be $\left(a,a^2\right)$.

$$\left.\frac{dy}{dx}\right|_{x=a}=\frac{a^2-0}{a-\frac{3}{4}}$$
- The slope of the tangent must be the same as the slope as calculated from the point on the curve and the outside point $\left(\frac{3}{4},0\right)$.

$$2a=\frac{a^2}{a-\frac{3}{4}}$$
- Simplify and move all terms to one side.

$$2a\left(a-\frac{3}{4}\right)=a^2$$

$$2a^2-\frac{3}{2}a=a^2$$

$$a^2-\frac{3}{2}a=0$$
- Factorize.

$$a\left(a-\frac{3}{2}\right)=0$$
- Note that while $a=0$ is a solution, it leads to a slope of $2a=0$. We want a non-horizontal tangent line, so we discard $a=0$.

$$a=\frac{3}{2}$$

$$y=\left.\frac{dy}{dx}\right|_{x=\frac{3}{2}}x+c$$
- Now we can write out the equation of the line.

$$=2\left(\frac{3}{2}\right)x+c$$
- To find the constant c, we can substitute in the point $\left(\frac{3}{4},0\right)$, which is on this line.

$$0=3\left(\frac{3}{4}\right)+c$$

$$c=-\frac{9}{4}$$

Therefore the line is $\underline{\underline{y=3x-\frac{9}{4}}}$

B5) Find $\frac{dy}{dx}$ if $y^3 + xy - 3y^2 - 2x - 2 = 0$. What is the value of $\frac{dy}{dx}$ when $x = 2$? [Chapter 3]

$$y^3 + xy - 3y^2 - 2x - 2 = 0$$

- Differentiate the equation implicitly.

$$\frac{d}{dx}\left(y^3 + xy - 3y^2 - 2x - 2\right) = \frac{d}{dx}0$$

- Chain Rule and Product Rule are both needed.

$$\underbrace{3y^2\frac{dy}{dx}}_{\text{Chain Rule}} + \underbrace{\frac{dx}{dx}y + x\frac{dy}{dx}}_{\text{Product Rule}} - \underbrace{6y\frac{dy}{dx}}_{\text{Chain Rule}} - 2 = 0$$

$$3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} - 6y\frac{dy}{dx} - 2 = 0$$

- Solve for $\frac{dy}{dx}$.

$$\left(3y^2 + x - 6y\right)\frac{dy}{dx} = 2 - y$$

$$\frac{dy}{dx} = \frac{2-y}{3y^2 + x - 6y}$$

- We now need to determine value of y that corresponds to $x = 2$.

$$y^3 + 2y - 3y^2 - 2(2) - 2 = 0$$

- Substitute $x = 2$ into the original equation.

$$y^3 - 3y^2 + 2y - 6 = 0$$

- $y = 3$ is a solution to this cubic equation, so $y - 3$ is a factor.

$$(y-3)\left(y^2 + 2\right) = 0$$

- The quadratic factor gives imaginary roots, which can be ignored.

$$y = 3$$

$$\left.\frac{dy}{dx}\right|_{x=2,y=3} = \frac{2-3}{3(3)^2 + 2 - 6(3)}$$

- We can now evaluate the derivative at $x = 2$.

$$= \frac{-1}{27 + 2 - 18}$$

$$= -\frac{1}{11}$$

B6) Suppose that *f* is a differentiable function such that $f'(8)=2$. Find the value of $\lim_{x\to 8}\frac{f(x)-f(8)}{x^{\frac{1}{3}}-2}$. [Chapter 3]

$$\lim_{x\to 8}\frac{f(x)-f(8)}{x^{\frac{1}{3}}-2}$$

- Since we're given a value for $f'(8)$ while asked to find the value of a quotient as $x\to 8$, let's try to write the expression we have into the form of the definition of the derivative: $f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$. We need to determine the factor that would bring the denominator to $x-8$, and multiply top and bottom by this factor.

$$=\lim_{x\to 8}\frac{f(x)-f(8)}{x^{\frac{1}{3}}-2}\cdot\frac{x^{\frac{2}{3}}+2x^{\frac{1}{3}}+4}{x^{\frac{2}{3}}+2x^{\frac{1}{3}}+4}$$

- $\left(x^{1/3}-2\right)\left(x^{2/3}+2x^{1/3}+4\right)=x-8$. The factor is determined term by term: $x^{2/3}$ is the factor required to get *x* from $x^{1/3}$, which then gives a $-2x^{2/3}$ that needs to be cancelled out by the 2nd term of the factor we seek. Repeating the process, the 2nd term is therefore $2x^{1/3}$, which leaves $-4x^{2/3}$ that needs to be cancelled out by the 3rd term. Finally, the 3rd term is 4 if we repeat the process one more time, which leaves -8 as we want.

$$=\lim_{x\to 8}\left[\frac{f(x)-f(8)}{x+8}\cdot\left(x^{\frac{2}{3}}+2x^{\frac{1}{3}}+4\right)\right]$$

- Now we can distribute the limit.

$$=\lim_{x\to 8}\frac{f(x)-f(8)}{x+8}\cdot\lim_{x\to 8}\left(x^{\frac{2}{3}}+2x^{\frac{1}{3}}+4\right)$$

- The 1st term is the derivative $f'(8)$ as we aimed for, while the 2nd term can be evaluated by substituting in $x=8$.

$$=f'(8)\cdot\left(8^{\frac{2}{3}}+2(8)^{\frac{1}{3}}+4\right)$$

$$=(4+4+4)f'(8)$$

$$=12f'(8)$$

- Substitute in $f'(8)=2$.

$$=12(2)$$

$$=\underline{\underline{24}}$$